Effect of anisotropy in ground movements caused by tunnelling

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This paper presents closed-form analytical solutions for estimating far-field ground deformations caused by shallow tunnelling in a linear elastic soil mass with cross-anisotropic stiffness properties. The solutions describe two-dimensional ground deformations for uniform convergence ($u_c$) and ovalisation ($u_o$) modes of a circular tunnel cavity, based on the complex formulation of planar elasticity and superposition of fundamental singularity solutions. The analyses are used to interpret measurements of ground deformations caused by open-face shield construction of a Jubilee Line Extension (JLE) tunnel in London Clay at a well-instrumented site in St James’s Park. Anisotropic stiffness parameters are estimated from hollow-cylinder tests on intact block samples of London Clay (from the Heathrow Airport Terminal 5 project), and the selection of the two input parameters is based on a least-squares optimisation using measurements of ground deformations. The results show consistent agreement with the measured distributions of surface and subsurface, vertical and horizontal displacement components, and anisotropic stiffness properties appear to have little effect on the pattern of ground movements. The results provide an interesting counterpoint to prior studies using finite-element analyses that have reported difficulties in predicting the distribution of ground movements for the instrumented section of the JLE tunnel.

KEYWORDS: anisotropy; elasticity; ground movements; settlement; theoretical analysis; tunnels

INTRODUCTION

All tunnel operations cause movements in the surrounding soil. Figs 1(a) and 1(b) illustrate the primary sources of movements for cases of closed-face shield tunnelling and open-face sequential support and excavation (often referred to as NATM) respectively. For open-face shield tunnelling, the stress changes around the tunnel face and the unsupported round length are primary sources of ground movements. Current geotechnical practice relies almost exclusively on empirical methods for estimating tunnel-induced ground deformations. Following Peck (1969) and Schmidt (1969), the transversal surface settlement trough can be fitted by a Gaussian function as

$$u_x(x, y) = u^0_y \exp \left( -\frac{x^2}{2\sigma^2} \right)$$

where $x$ is the horizontal distance from the tunnel centreline, $u^0_y$ is the surface settlement at the tunnel centreline, and $x_i$ is the location of the inflection point.

Mair & Taylor (1997) show that the width of the settlement trough is well correlated with the depth of the tunnel, $H$, and with the characteristics of the overlying soil (see Fig. 2(a)). The same framework has been extended to subsurface vertical movements by varying the trough width parameter to give

$$x_i = K(H - y)$$

where $K$ is the non-linear function shown in Fig. 2(b).

There are very limited data for estimating the horizontal components of ground deformations. The most commonly used interpretation is to assume that the displacement vectors are directed to a point on or close to the centre of the tunnel, as proposed by Attewell (1978) and O’Reilly & New (1982), such that

$$u_x \approx \left( \frac{x}{H} \right) u_y$$

There are also a variety of analytical solutions that have been proposed for estimating the two-dimensional distributions of ground movements for shallow tunnels. These analyses make simplifying assumptions regarding the constitutive behaviour of soil, and ignore details of the tunnel construction procedure, but otherwise fulfil the principles of continuum mechanics. In principle, these analytical solutions provide a more consistent framework for interpreting horizontal and vertical components of ground deformations than conventional empirical models, and use a small number of input parameters that can be readily calibrated to field data. They also provide a useful basis for evaluating the accuracy of more complex numerical analyses. However, the analytical solutions do not purport to describe the processes of tunnel construction accurately, and hence are limited to estimation of far-field ground deformations. In contrast, more comprehensive finite-element (FE) analyses are also able to compute near-field behaviour (such as stresses in tunnel lining systems), and hence offer a complete predictive framework for simulating tunnel construction processes and their effects on adjacent structures.

The ‘far-field’ ground movements caused by shallow tunnelling processes (excavation and support) are solved as a linear combination of deformation modes occurring at the tunnel cavity (Fig. 3), with input parameters, $u_c$ and $u_o$, corresponding to uniform convergence and ovalisation respectively. Pinto & Whittle (2013) have shown that closed-form solutions obtained by superposition of singularity solutions (after Sagaseta, 1987) provide a good approximation to the more complete (‘exact’) solutions obtained by representing the finite dimensions of a shallow tunnel in a linear elastic soil (after Verruijt & Booker, 1996; Verruijt, 1997).

Pinto et al. (2013) have evaluated the analytical solutions through a series of case studies involving tunnels excavated...
through different ground conditions using a variety of closed- and open-face construction methods. They generally found good agreement with measured data for tunnels constructed in low-permeability clays, assuming isotropic elastic properties. Although the analytical solutions do not simulate the actual tunnel construction process, the effects of changes in control parameters (such as the face pressure in earth pressure balance (EPB) tunnelling) will affect far-field ground movements, and will be reflected in changes in the cavity deformation parameters $u/C_{229}$ and $u/C_{228}$.

Pinto et al. (2013) noted significant limitations for the case of the Heathrow Express trial tunnel (Deane & Bassett, 1995), and the discrepancies between predicted and measured settlements were attributed to anisotropic stiffness properties of the heavily overconsolidated London Clay.

More recently, Gasparre et al. (2007) have presented results from a comprehensive and definitive laboratory investigation of the stiffness properties of natural London Clay using block samples obtained during the excavations for Heathrow Terminal 5. Their test programme included measurements of small-strain elastic properties (based principally on wave propagation data using triaxial devices equipped with bender elements), limits on the reversible elastic response (referred to as the $Y_1$ yield condition) through drained and undrained triaxial stress probe tests, and measurements of the degradation of secant stiffness parameters with strain level (using local strain measurements in triaxial and hollow-cylinder devices). They conclude that the small-strain behaviour of the clay is well described by the framework of cross-anisotropic elasticity, and that ‘significant anisotropy was revealed at all scales of deformation’.

This paper extends the analytical solutions presented by Pinto & Whittle (2013) to account for cross-anisotropic stiffness properties of the clay. The solutions are then evaluated through comparisons with data from the Jubilee Line Extension (JLE) project, involving open-face shield tunnel construction beneath a well-instrumented site in St James’s Park (Nyren, 1998). This is a very well-instrumented and documented case site, with extensive supporting data on cross-anisotropic stiffness parameters for London Clay reported by Gasparre et al. (2007). The JLE test section has been extensively analysed by others using FE analyses, many have reported problems in predicting far-field deformations, and hence it provides an interesting opportunity to assess the capabilities of the proposed analytical solutions. Independent research by Puzrin et al. (2012) has attempted to model the same case study using a related analytical approach.

**ANALYTICAL SOLUTIONS FOR CROSS-ANISOTROPIC ELASTIC SOIL**

The current analyses consider deformations in a vertical plane ($x, y$) through a cross-anisotropic, linear elastic soil with isotropic properties in a plane with dip angle $\alpha$ to the horizontal, as shown in Fig. 4. The stiffness parameters of the soil are given for a local $(x', y', z')$ coordinate system (Appendix 1 shows the transformation to the global frame $(x, y, z)$). The five independent anisotropic stiffness parameters are defined in the local coordinate system as $E_{1i}$, the
Young’s modulus of the soil in a direction parallel to the isotropic plane; \( E_1 \), the Poisson’s ratio of strains in the isotropic plane (\( x' - z' \)); \( E_2 \), the Young’s modulus normal to the isotropic plane; \( G_2 \), the shear modulus for strain in direction \( y' \); and \( \nu_2 \), the Poisson’s ratio for strain in the \( y' \) direction due to strain in the \( x' \) direction.

Following Milne-Thompson (1960) and Lekhnitskii (1963) the stress–strain relations for plane-strain geometry conditions can be written as

\[
\begin{align*}
\sigma_{xx} & = \\
\sigma_{yy} & = \\
\tau_{xy} & =
\end{align*}
\]

\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix}
= 
\begin{bmatrix}
\beta_{11} & \beta_{12} & \beta_{16} \\
\beta_{12} & \beta_{22} & \beta_{26} \\
\beta_{16} & \beta_{26} & \beta_{66}
\end{bmatrix}
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix}
\tag{4a}
\]

where the \( \beta_{ij} \) coefficients are related to the five independent stiffness parameters and the dip angle \( \alpha \), as shown in Table 1. For \( \alpha = 0^\circ \) (i.e. isotropic properties in the horizontal plane), \( E_i = E_h, \nu_1 = \nu_{hh}, E_2 = E_v, \nu_2 = \nu_{hv}, G_2 = G_{hv} \), and the \( \beta_{ij} \) coefficients are
show that London Clay is strongly anisotropic at very small strain levels (true elastic range). The stiffness ratio, \( n = E_v / E_h \), varies only slightly \((n = 1.72–2.30)\), while \( m = G_{bh}/E_v \) increases from 0.66 to 1.27 with increased strain level. The small-strain stiffness ratios calculated from undrained tests are very similar to those from drained parameters, as shown in Fig. 5.

The elastic parameters are further constrained by thermodynamic considerations (e.g. Pickering, 1970), such that

\[
G_{bh}, \ E_v, \ E_h > 0 \quad 0 < n < 4
\]

\[
-1 < n_{bh} < 1
\]

\[
n_{bh} + 2n_{bh}n_{bh} \leq 1
\]

The conditions for incompressibility are given by Gibson (1974) as

\[
\beta_1 = 1 - \frac{v_{bh}^2}{E_h}
\]

\[
\beta_{12} = -\frac{v_{bh}}{E_v}(1 + v_{bh})
\]

\[
\beta_{22} = \frac{1}{E_v}(1 - v_{bh}^2)
\]

\[
\beta_6 = \frac{1}{G_{bh}}
\]

\[
\beta_{16} = \beta_{26} = 0
\]

where the stiffness ratios \( n = E_h/E_v \) and \( m = G_{bh}/E_v \) are used later in the paper.

Figure 5 shows non-linear secant stiffness measurements of \( E_v, \ E_h \) and \( G_{bh} \) from drained, hollow-cylinder (HCA), uniaxial load tests on natural London Clay (unit B2) as functions of strain level (Gasparre et al., 2007). The data

\[
G_{bh}, \ E_v, \ E_h > 0
\]

\[
0 < n < 4
\]

\[
-1 < n_{bh} < 1
\]

\[
n_{bh} + 2n_{bh}n_{bh} \leq 1
\]

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\[
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\beta_{12} = -\frac{v_{bh}}{E_v}(1 + v_{bh})
\]

\[
\beta_{22} = \frac{1}{E_v}(1 - v_{bh}^2)
\]

\[
\beta_6 = \frac{1}{G_{bh}}
\]

\[
\beta_{16} = \beta_{26} = 0
\]
\[ \nu_{bh} = 0.5 \]

\[ \nu_{bh} = 1 - 2n^2 = 1 - \frac{n}{2} \]

In the absence of body forces, the stresses can be solved using the Airy stress function, \( F(x, y) \), to give

\[
\begin{align*}
\beta_{22} \frac{\partial^2 F}{\partial x^2} - 2\beta_{26} \frac{\partial^2 F}{\partial x \partial y} + (2\beta_{12} + \beta_{66}) \frac{\partial^4 F}{\partial x^2 \partial y^2} \\
- 2\beta_{16} \frac{\partial^2 F}{\partial x \partial y^3} + \beta_{11} \frac{\partial^4 F}{\partial y^4} = 0
\end{align*}
\] (4d)

The roots of this equation are conjugate complex numbers, say \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \), and without loss of generality \( \lambda_1 = a_1 + ib_1, \lambda_2 = a_2 + ib_2 \), and \( b_1 > b_2 > 0 \). Any analytic function \( g(x + iy) \) satisfies equation (5) if \( \lambda \) is a solution to the characteristic equation. Since the resulting stress function must be real, the general solution is given by the expression

\[
F(x, y) = 2\text{Re}[F_1(z_1) + F_2(z_2)]
\]

\[
= F_1(z_1) + F_1(z_1) + F_2(z_2) + F_2(z_2)
\] (7)

where \( z_1 = x + \lambda_1 y, z_2 = x + \lambda_2 y \). Introducing the new functions \( \Phi_1(z_1) = F_1(z_1) \), the stresses are found using the definition of complex variables \( z_1, z_2 \) as

\[
\begin{align*}
\sigma_x &= 2\text{Re}[\lambda_1 \Phi_1(z_1) + \lambda_2 \Phi_2(z_2)] \\
\sigma_y &= 2\text{Re}[\Phi_1(z_1) + \Phi_2(z_2)] \\
\tau_{xy} &= -2\text{Re}[\lambda_1 \Phi_1(z_1) + \lambda_2 \Phi_2(z_2)]
\end{align*}
\] (8a)

(8b)

(8c)

and the displacements \( U(x, y) \), \( V(x, y) \) are found by integrating the strains, to give

\[
U = 2\text{Re}[p_1 \Phi_1(z_1) + p_2 \Phi_2(z_2)]
\]

\[
V = 2\text{Re}[q_1 \Phi_1(z_1) + q_2 \Phi_2(z_2)]
\] (9a)

(9b)

where the coefficients \( p_k, q_k \) are expressed as

\[
p_k = \beta_{11} \lambda_k^2 + \beta_{12} - \beta_{16} \lambda_k
\]

\[
q_k = \beta_{12} \lambda_k + \beta_{22} \lambda_k - \beta_{26}
\] (10)

\[ k = 1, 2 \]
Uniform convergence mode
For a cylindrical cavity of radius $R$ in an infinite medium undergoing uniform convergence, $u_\epsilon$, the displacement components at the tunnel wall can be expressed by (Fig. 6(a))

\begin{align}
\varphi_0(\theta) &= u_\epsilon \cos \theta \\
&= \frac{e^{i\theta} + e^{-i\theta}}{2} \\
&= u_\epsilon \left( \frac{\alpha + \alpha^{-1}}{2} \right) \\
\psi_0(\theta) &= u_\epsilon \sin \theta \\
&= \frac{e^{i\theta} - e^{-i\theta}}{2i} \\
&= u_\epsilon \left( \frac{\alpha - \alpha^{-1}}{2i} \right) \\
\end{align}

(11a)

(11b)

where $\alpha = e^{i\theta}$.

The circular boundary of the tunnel cavity in the $(x, y)$ plane is transformed into an inclined ellipse in the plane of the complex variable $z = x + iy = Re^{i\theta} = R\zeta$ (Fig. 6(b)).

\begin{align}
z_1 &= x + \lambda_1 y \\
&= x + \Re\{\lambda_1\} y + i\Im\{\lambda_1\} y \\
&= x_1 + iy_1 \\
\end{align}

(12a)

\begin{align}
z_2 &= x + \lambda_2 y \\
&= x + \Re\{\lambda_2\} y + i\Im\{\lambda_2\} y \\
&= x_2 + iy_2 \\
\end{align}

(12b)

The boundary conditions can be solved by a further mapping onto a circle of unit radius, as shown in Fig. 6(b).

\begin{align}
\zeta_k &= R \left( \frac{1 - i\lambda_k}{2} \zeta_k + \frac{1 + i\lambda_k}{2} \right) \\
&= \zeta_k \pm \frac{z_k - R^2 \left( 1 + \lambda_k^2 \right)^{1/2}}{R(1 - i\lambda_k)} \\
\end{align}

(13)

where $k = 1, 2, |\zeta_k| > 1$.

The analytic functions $\Phi_k(z_k)$ can be expressed as a Laurent series of the conformed variable $\zeta_k$.

\begin{align}
\Phi_1(z_1) &= \Phi_1(\zeta_1) \\
&= \sum_{n=0}^\infty a_n \zeta_1^{-n} \\
\Phi_2(z_2) &= \Phi_2(\zeta_2) \\
&= \sum_{n=0}^\infty b_n \zeta_2^{-n} \\
\end{align}

(14a)

(14b)

Fig. 6. (a) Prescribed displacement modes at tunnel cavity; (b) problem boundaries in $z_k$-plane and in transformed plane
At the tunnel wall, \( z = R \) and \( \zeta_2 = e^{i\theta} = \sigma \). Hence, from equation (9), the displacement components can be found from

\[
\begin{align*}
p_1 \sum_{n=0}^{\infty} a_n \alpha^{-n} + P_1 \sum_{n=0}^{\infty} b_n \alpha^{-n} + P_2 \sum_{n=0}^{\infty} b_n \alpha^{-n} + q_1 \sum_{n=0}^{\infty} a_n \alpha^{-n} + q_1 \sum_{n=0}^{\infty} a_n \alpha^{-n} + q_2 \sum_{n=0}^{\infty} b_n \alpha^{-n} + q_2 \sum_{n=0}^{\infty} b_n \alpha^{-n} + \alpha^{-1} \frac{\alpha - \sigma^{-1}}{\sigma} \\
\end{align*}
\]

Equating the coefficients for powers of \( \sigma \)

\[
\begin{align*}
p_1 a_1 + p_2 b_1 &= \frac{u_z}{2} \\
q_1 a_1 + q_2 b_1 &= \frac{iu_z}{2} \\
n \neq 1: \\
p_1 a_n + p_2 b_n &= 0 \\
q_1 a_n + q_2 b_n &= 0 \\
\end{align*}
\]

The series coefficients are then solved as

\[
\begin{align*}
a_1 &= \frac{u_z}{2} \left( \frac{q_2 - ip_2}{p_1 q_2 - q_1 p_2} \right) \\
b_1 &= \frac{u_z}{2} \left( \frac{-q_1 + ip_1}{p_1 q_2 - q_1 p_2} \right) \\
n \neq 1: \\
a_n &= b_n = 0 \\
\end{align*}
\]

Ovalisation mode

The ovalisation mode involves no ground loss, and displacements at the tunnel cavity can be represented as follows.

\[
\begin{align*}
u_x(\theta) &= u_\delta \cos \theta \\
&= u_\delta \left( e^{i\theta} + e^{-i\theta} \right) / 2 \\
v_x(\theta) &= -u_\delta \sin \theta \\
&= -u_\delta \left( e^{i\theta} - e^{-i\theta} \right) / 2i \\
\end{align*}
\]

Applying the same methodology used above (for uniform convergence) the series coefficients \( a_n, b_n \) are found.

\[
\begin{align*}
an = 1: \\
a_1 &= \frac{u_0}{2} \left( \frac{q_2 + ip_2}{p_1 q_2 - q_1 p_2} \right) \\
b_1 &= \frac{u_0}{2} \left( \frac{-q_1 - ip_1}{p_1 q_2 - q_1 p_2} \right) \\
n \neq 1: \\
a_n &= b_n = 0 \\
\end{align*}
\]

Effect of traction-free ground surface

Following Sagaseta (1987), the ground movements associated with a shallow tunnel located at a depth \( H \) below the traction-free ground surface can be represented approximately through a singularity superposition technique (Fig. 4). The deformation field for the shallow tunnel is represented by superimposing full-space solutions for a point source \((0, -H)\) and mirror image sink \((0, +H)\) (i.e. with equal and opposite cavity deformations) relative to the stress-free ground surface \((y = 0)\) respectively.

Contracting tunnel \((-u_z > 0, u_0 > 0)\):

\[
u^c_x(x, y) = U(x, y) + V(x, y) + \frac{1}{2} U_0(x, y) + \frac{1}{2} V_0(x, y) + u_0 x \\
\]

Mirror image \((-u_z < 0, u_0 < 0)\):

\[
u^s_x(x, y) = -U(x, y) + V(x, y) + \frac{1}{2} U_0(x, y) + \frac{1}{2} V_0(x, y) - u_0 x \\
\]

The resulting normal and shear tractions at the surface \( y = 0 \) due to these mirror images are as follows.

\[
\begin{align*}
N^c(x) &= \sigma_y^c(x, 0) + \sigma_y^s(x, 0) \\
&= \sigma_y(x, H) - \sigma_y(x, -H) \\
&= 0 \\
T^c(x) &= t_{xy}^c(x, 0) + t_{xy}^s(x, 0) \\
&= t_{xy}(x, H) - t_{xy}(x, -H) \\
&= -2t_{xy}(x, -H) \\
\end{align*}
\]

A set of (equal and opposite) ‘corrective’ shear tractions \( T^c(x) \) must then be applied at the free surface (Fig. 4). The resulting displacements on a half-plane due to these corrective stresses are

\[
\begin{align*}
u^c_x &= 2Re\left[ p_1 \Phi^c(z) + q_1 \Phi^s(z) \right] \\
u^s_x &= 2Re\left[ q_1 \Phi^c(z) + q_2 \Phi^s(z) \right] \\
\end{align*}
\]

where the analytic functions \( \Phi^c, \Phi^s \) are obtained through integration (after Lekhnitskii, 1963).
and the integrals of the normal and shear tractions along the boundary are

\[
\Phi_1(z_1) = \frac{1}{\lambda_1 - \lambda_2} \int_{-\infty}^{\infty} \frac{\lambda_2 f_1(\xi) + f_2(\xi)}{\xi - z_1} d\xi
\]

(25a)

\[
\Phi_2(z_2) = -\frac{1}{\lambda_1 - \lambda_2} \int_{-\infty}^{\infty} \frac{\lambda_1 f_1(\xi) + f_2(\xi)}{\xi - z_2} d\xi
\]

(25b)

The final field of ground deformations for a shallow tunnel with uniform convergence at the tunnel cavity is then obtained from equations (21), (22) and (24).

\[
u_s(x, y) = u_s^1(x, y) + u_s^2(x, y) + u_s^c(x, y)
\]

(27a)

\[
u_u(x, y) = u_u^1(x, y) + u_u^2(x, y) + u_u^c(x, y)
\]

(27b)

Typical results

Figures 7 and 8 illustrate the effects of cross-anisotropic stiffness properties on predictions of the shape of the surface settlement trough and lateral deflections for a shallow tunnel in clay with \( R/H = 0.22 \) (and typical cross-anisotropic stiffness ratios, \( n \) and \( m \)). Fig. 7 shows that for horizontal planes of isotropy \( (\alpha = 0^\circ) \), as the ovalisation ratio \( \rho = -u_3/u_2 \) increases, the predicted settlement troughs become narrower and the surface centreline settlement, \( u_{sy} \), increases significantly. For \( \rho = 0 \) the analyses predict inward horizontal displacements near the tunnel springline, while increases in \( \rho \) result in larger outward movements at this elevation (Fig. 7(b)). Fig. 7 also illustrates results for cases where the plane of isotropy is dipping \( (\alpha = 0^\circ, 15^\circ, 30^\circ, 45^\circ \) and \( \rho = 1 \)), representing (for example) conditions at the edge of a

![Fig. 7. Effect of relative distortion and dip angle on predicted surface settlements and subsurface lateral displacements for cross-anisotropic clay](image-url)
Fig. 8. Effect of anisotropic stiffness ratios \((n \text{ and } m)\) on predicted surface settlements and subsurface lateral displacements: (a) normalised surface settlement trough; (b) normalised lateral displacements at offset, \(x/2R = 1\)

Prior Interpretation of JLE Tunnel in St James’s Park

The JLE project (1994–1999) included 15 km of twin bored tunnels, 4.85 m in diameter, constructed using an open-face shield and excavated by mechanical backhoe. Ground displacements were measured at a well-instrumented greenfield site in St James’s Park, and were described in detail by Nyren (1998). The westbound (WB) tunnel passed under the instrumentation site in April 1995 with springline depth \(H = 31\) m and an advance rate of 45.5 m/day (i.e. 1.9 m/h). The eastbound (EB) tunnel (not considered in this paper) traversed the section in January 1996 at depth \(H = 20.5\) m (and offset at 21.5 m from the WB bore).

The instrumentation at the test section included an array of 24 surface monitoring points (SMP; surveyed by total stations), and subsurface ground movements were recorded using a set of: (a) nineelectrolevel inclinometers, with tilt angles typically measured at vertical intervals of 2.5 m; and (b) 11 rod extensometers, each measuring vertical displacement components at up to eight elevations. Fig. 10 shows eight locations (A–H) where two-dimensional vectors of displacement can be interpreted from the inclinometer and extensometer data.

The soil profile comprises 12 m of fill, alluvium and terrace gravels overlying a 40 m thick unit of low-permeability sediments. As the dip angle decreases, the predicted surface centreline settlement \(u_{xy}^0\) increases, while the effect on the horizontal displacements is less pronounced.

Figure 8 shows the effects of the stiffness ratios \(n\) and \(m\) for the case where the soil has isotropic properties in the horizontal plane \((\alpha = 0^\circ)\). The results show a narrowing of the surface settlement trough for normal stiffness ratios, \(n > 1\) \((n = E_h/E_v = 1\) is the isotropic case), which is especially pronounced for \(n > 3\). Increases in the shear stiffness ratio, \(m = G_h/E_v\), have the opposite effect. The settlement trough for \(m = 1\) is much wider than the isotropic case \((m = 0.33)\). There is also a change in the mode shape of the settlement trough shown for \(m = 1.5\), where the maximum settlement does not occur above the centreline, but is instead offset at \(x/H \approx 0.5\). This result is often observed in two-dimensional FE analyses of shallow tunnel excavation for cases with high in situ \(K_0\) stress conditions (e.g. Addenbrooke et al., 1997; Franzius et al., 2005; Möller, 2006), but has not been reported in prior tunnelling projects. The transition in mode shape is a function of the anisotropic stiffness ratios \((m \text{ and } n)\) and the ovalisation ratio, \(\rho\), as shown in Fig. 9. The subsequent applications of the analyses for the JLE tunnel use a constrained range of \(\rho\) to avoid the higher mode solutions.
London Clay (with four divisions shown in Fig. 10), above the Lambeth Group (lower aquifer system). The groundwater table is located approximately 3 m below the ground surface, but pore pressures are 5–7 m below hydrostatic at the elevation of the WB tunnel springline. Standing & Burland (2006) have carried out a detailed review of the physical and engineering properties of the four divisions of the London Clay along this section of the JLE alignment. They report the undrained shear strength of London Clay, \( s_u \), increasing from \( 215 \pm 80 \) kPa (unit A3) to \( 233 \pm 77 \) kPa (A2), and in situ hydraulic conductivity values, \( k = 0.15–2 \times 10^{-10} \) m/s.

Surface displacements

Figures 11(a) and 11(b) show the vertical and horizontal surface movements measured approximately 1 day after the passage of the WB tunnel face, when it can reasonably be assumed that there is little consolidation within the London Clay.
Clay. Standing & Burland (2006) fitted the transverse surface settlement trough using the empirical Gaussian relation (equation (1)) with a trough width, $\chi = 13.3$ m (i.e. $K = x_0/H = 0.43$) and maximum settlement above the crown, $u^0_H \approx 20$ mm. Hence the volume loss at the ground surface, $\Delta V_x (= 2.5u_0m)$ corresponds to an apparent ground loss at the tunnel cavity, $\Delta V / V_0 = 3.3\%$, caused by tunnel construction. They attribute this unexpectedly high volume loss to a deviation in principal stresses acting in the theoretical elastic range of $n$; equation 4(c)) in combination with a low value of $K_0 = 0.5$. However, when the same model parameters were used in a three-dimensional analysis of the open-face tunnel construction, much larger surface settlements were obtained ($u^0_H = 85$ mm with interpreted volume loss, $\Delta V / V_0 = 18\%$), as shown in Fig. 12.

Prior numerical analyses

Several researchers have attempted to compute the ground movements reported by Nyren (1998) using non-linear FE methods. For example, Franzius et al. (2005) compared two-dimensional and three-dimensional analyses using different coefficients of lateral earth pressure at rest, $K_0$, and various constitutive models for simulating the construction of the JLE WB tunnel. Their base-case scenario used a non-linear, isotropic elasto-plastic constitutive model with $K_0 = 1.5$. In two-dimensional analyses they assumed a volume loss $\Delta V / V_0 = 3.3\%$, which resulted in a computed maximum surface settlement $u^0_H = 10$ mm and a transverse surface settlement trough that was much wider than the measured behaviour (Fig. 12). Surprisingly, they also found similar results from three-dimensional analyses using a step-by-step procedure that simulates the boundary conditions associated with open-face excavation and lining construction.

Franzius et al. (2005) then modified the constitutive model to include non-linear cross-anisotropic stiffness properties (using a simplified three-parameter formulation proposed by Graham & Houlsby, 1983). They were able to achieve good agreement with the measured settlement trough in the two-dimensional analyses only by using an unrealistically high elastic Young’s modulus ratio $n = E_y/E_v = 6.5$ (i.e. outside the theoretical elastic range of $n$; equation 4(c)) in combination with a low value of $K_0 = 0.5$. However, when the same model parameters were used in a three-dimensional analysis of the open-face tunnel construction, much larger surface settlements were obtained ($u^0_H = 85$ mm with interpreted volume loss, $\Delta V / V_0 = 18\%$), as shown in Fig. 12.

Wongsaroj (2005) formulated a bespoke constitutive model to describe the non-linear, anisotropic behaviour of London Clay, and used the model in three-dimensional FE simulations for short- and long-term ground movements caused by JLE tunnel construction. Fig. 13(a) compares the measured surface settlements with computed results using four different input parameter sets. Models with both iso-

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**Fig. 11.** Empirical interpretation of surface displacements for WB JLE tunnel in St James’s Park: (a) surface settlement trough (after Standing & Burland, 2006); (b) empirical interpretation of surface displacements for WB JLE tunnel in St James’s Park

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### Table

<table>
<thead>
<tr>
<th>$\Delta V / V_0$</th>
<th>$\chi$</th>
<th>$m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.3%</td>
<td>13.3</td>
<td></td>
</tr>
</tbody>
</table>
tropic and anisotropic small-strain stiffness ($K_0 = 1.5$; small-strain, drained elastic stiffness ratios, $n = 0.44$, $m = 0.13$, that are inconsistent with data shown in Fig. 5) resulted in settlement troughs that are wider than the field measurements for the WB JLE tunnel, and also significantly overestimate the back-figured volume loss ($\Delta V/L = 5.4–6.0\%$). Good agreement is achieved only by increasing the anisotropic stiffness ratio ($G_{hh}/G_{vh} = 5$, corresponding to $m = 0.04$) and reducing the assumed value of $K_0 = 1.2$. Fig. 13(b) shows further comparisons with the subsurface horizontal displacements reported by Wongsaroj (2005). The analyses generally predict larger lateral deformations of the soil towards the tunnel centreline than are measured in the field. The author attributed this discrepancy, in part, to surveying errors in the field measurements. Subsurface horizontal displacements were not reported for the fourth model ($G_{hh}/G_{vh} = 5$), and thus are not shown in Fig. 13(b).

### APPLICATION OF ANALYTICAL SOLUTIONS

In contrast to the preceding analyses, which are based on comprehensive three-dimensional FE analyses, the proposed analytical solutions make simplifying constitutive assumptions in order to solve the two-dimensional far-field ground deformations as functions of the two cavity deformation parameters, $u_c$ and $u_k$. These parameters are back-fitted from the measured deformations of the WB JLE tunnel in St James's Park using a least-squares fitting approach. The current analyses assume linear elastic behaviour throughout the soil mass, and hence are likely to underestimate ground deformations close to the tunnel lining, where plastic failure occurs in the clay.

This near-field zone of plasticity can be estimated from solutions of a cylindrical cavity contraction in an elastic-perfectly plastic soil (e.g. Yu & Rowe, 1999). The radius of the plastic zone, $R_p$, can be found from

$$\frac{R_p}{R} = \exp \left( \frac{N - 1}{2} \right)$$

where $N = (p_0 - p_t)/\sigma_u$ is the overload factor, and $p_0$ and $p_t$ are the pressures in the far field and within the tunnel cavity.

The radius of the plastic zone can then be estimated by

(a) equating $p_0$ with the overburden pressure ($\sigma_{u0} \approx 600$ kPa) at the springline

(b) assuming $p_t = 0$

(c) considering a likely range of undrained shear strength for
### Table: 3D finite-element analysis results, after Wongsaroj (2005)

<table>
<thead>
<tr>
<th>Line</th>
<th>Model</th>
<th>$K_0$</th>
<th>$G_m/G_m$</th>
<th>$\Delta V_c/V_c$, %</th>
<th>$n$</th>
<th>$m$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Isotropic</td>
<td>1.5</td>
<td>1.0</td>
<td>6.0</td>
<td>1.000</td>
<td>0.435</td>
</tr>
<tr>
<td></td>
<td>Anisotropic</td>
<td>1.5</td>
<td>1.5</td>
<td>5.6</td>
<td>0.438</td>
<td>0.130</td>
</tr>
<tr>
<td></td>
<td>Anisotropic</td>
<td>1.0</td>
<td>1.5</td>
<td>5.4</td>
<td>0.438</td>
<td>0.130</td>
</tr>
<tr>
<td></td>
<td>Anisotropic</td>
<td>1.2</td>
<td>5.0</td>
<td>3.2</td>
<td>0.438</td>
<td>0.039</td>
</tr>
</tbody>
</table>

Field measurements

Poisson’s ratios: isotropic ($\nu_{nh} = \nu_{hv} = \nu_{vn} = 0.15$);
anisotropic model ($\nu_{nh} = 0.07$, $\nu_{hv} = 0.12$, $\nu_{vn} = 0.16$)

**Fig. 13.** Comparison between field measurements and FE analysis results undertaken by Wongsaroj (2005): (a) surface settlement trough; (b) subsurface lateral displacements
the London Clay, $s_u = 136–293$ kPa (A3 unit; Standing & Burland, 2006).

Based on these assumptions, $R_p \approx 4–13$ m. The current interpretation excludes measured data within the estimated plastic zone, but considers 49 subsurface deformations (along eight vertical lines, A–H, Fig. 10), together with 24 locations where surface movements were surveyed. Fig. 14 shows the derivation of the least-squares solution error (LSS) for the input parameter state space $(u_i, \delta_i)$, where

$$LSS = \text{Min} \sum_i \left[ (\tilde{u}_{i,j} - u_{i,j})^2 + (\tilde{u}_{i,j} - u_{i,j})^2 \right]$$

(29)

In most practical cases, engineers will expect to fit the measured centreline surface settlement, $u_{c1}$, and hence it is preferable to present a modified least-squares solution, LSS*, that includes this additional constraint.

Figures 14(a) and 14(b) compare results for two sets of soil stiffness properties: (a) the isotropic case ($m = 0.33$, $n = 1$, $r = v_{sh} = v_{sh} = 0.5$), and (b) the cross-anisotropic case (with $\alpha = 0$), based on the small-strain behaviour reported by Gasparre et al. (2007), and assuming incompressibility of the London Clay ($m = 0.66$, $n = 2.07$, $v_{sh} = 0.5$, $v_{sh} = 1 - 0.5n = -0.035$). It should be noted that the small-strain elastic anisotropic stiffness ratio, $n = E_{hi}/E_{si}$, obtained from undrained tests is very close to that obtained from drained tests, as shown in Gasparre et al. (2007).

There is little difference in the magnitude of the global least-squares error between the two sets of analyses, while the constrained LSS* solution for the isotropic case is slightly closer to the global minimum than the cross-anisotropic case. The derived cavity contraction parameter is smaller for the cross-anisotropic case ($-\delta_i = 34$ mm, compared with 36 mm for the isotropic case), with a higher relative distortion, $\rho = -\delta_i/u_i = 1.56$ compared with 1.32. Both LSS* solutions imply slightly lower volume loss ratios at the tunnel cavity ($\Delta V_i/V_i = 3\%$ and $2.8\%$; Figs 14(a) and 14(b) respectively) than were estimated by conventional empirical solutions (3–3.3%; Fig. 11(a)).

Figure 15 compares analytical solutions of the distributions of vertical and horizontal surface displacement components for the WB JLE tunnel, using isotropic and cross-anisotropic soil properties (with LSS* tunnel mode input parameters). The fields of vertical displacements are very similar for both sets of analyses, while the cross-anisotropic case predicts slightly larger lateral ground movements around the tunnel springline than the isotropic case (Fig. 15(b)).

Figures 16(a) and 16(b) show that both sets of analyses produce very reasonable agreement with the measured vertical and horizontal surface displacements. These results show that reasonable predictions of surface displacements can be achieved using the analytical solutions with isotropic stiffness properties for the London Clay. This is a very surprising result, which is due to the counteracting effects of the two key stiffness ratios, $n$ and $m$ (compare Figs 8(a) and 8(b)).

Figures 17(a) and 17(b) compare the computed and measured subsurface vertical and horizontal displacement components for the WB JLE tunnel. The computed deformations are generally in very good agreement with both vertical and horizontal components of movements measured in the far field (i.e. outside the expected zone of plastic soil behaviour). Very similar patterns of soil displacements are obtained using isotropic and anisotropic elastic stiffness parameters. The analysis tends to overestimate measured centreline vertical settlements below 10 m, but produces very accurate predictions at the rest of the extensometer positions. The analytical solutions fit well the inclinometer readings at locations from the ground surface up to a transition depth marked by contour line $u_i = 0$ mm in Fig. 17(b), but predict outward movement below this transition depth, while the inclinometers show zero ground movements.

**CONCLUSIONS**

This paper has presented new analytical solutions for estimating two-dimensional ground deformations caused by
Fig. 15. Analytical predictions of vertical and horizontal ground deformations for LSS$^5$ solutions with isotropic and cross-anisotropic stiffness properties for London Clay: (a) vertical displacements (mm); (b) horizontal displacements (mm)
shallow tunnelling in a cross-anisotropic soil. These analyses extend prior solutions derived by Pinto (1999), Whittle & Sagaseta (2003) and Pinto & Whittle (2013) in which the complete distribution of far-field ground movements can be interpreted from two basic tunnel cavity deformation mode parameters ($u_c$ and $u_d$ or $\rho$), the dip angle of the isotropic stiffness plane, $\alpha$, and two key anisotropic stiffness ratios, $n = E_h/E_v$ and $m = G_{vh}/E_v$.

The analytical solutions have been applied to reinterpret ground deformations associated with the open-face construction of the WB tunnel for the Jubilee Line at a well-instrumented site in St James’ Park (Nyren, 1998). The current analyses benefit from high-quality measurements of the cross-anisotropic stiffness properties of intact London Clay measured in an independent study for Heathrow Airport T5 (Gasparre et al., 2007). These data show that London Clay exhibits pronounced stiffness anisotropy at small strain levels.

The cavity deformation mode parameters are evaluated using a least-squares fit to surface and subsurface deformations at the instrumented test site. The results show that both the isotropic and cross-anisotropic analytical solutions produce very good fits to the measured ground displacements. Using the high-quality measurements undertaken by Gasparre et al. (2007), it can indeed be concluded that cross-anisotropic stiffness parameters have only a small influence on predictions of the far-field ground deformations caused by tunnelling in London Clay. The analytical solutions achieve comparable levels of agreement with measurements of the surface settlement trough that are conventionally fitted using an empirical Gaussian distribution function. However, the current analytical solutions correspond to smaller volume losses at the tunnel cavity than those estimated by conventional empirical assumptions (cf. Standing & Burland, 2006), while offering a more consistent framework for interpreting the complete distribution of horizontal and vertical components of ground deformations. Although these results are very encouraging, further case studies are needed to establish how the cavity mode parameters are related to different methods of tunnel construction.

<table>
<thead>
<tr>
<th>Line</th>
<th>Analytical model</th>
<th>$u_c$; mm</th>
<th>$\Delta V/v_0$; %</th>
<th>$\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Isotropic</td>
<td>36-0</td>
<td>3-0</td>
<td>1-32</td>
<td></td>
</tr>
<tr>
<td>Anisotropic</td>
<td>34-0</td>
<td>2-8</td>
<td>1-56</td>
<td></td>
</tr>
</tbody>
</table>

*Fig. 16. Comparison of computed and measured surface movements for WB JLE tunnel: (a) settlements; (b) horizontal displacements*
Fig. 17. Comparison of computed and measured subsurface ground movements for WB JLE tunnel: (a) vertical displacements; (b) horizontal displacements
ACKNOWLEDGEMENTS
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APPENDIX 1: ROTATION OF PLANES FROM LOCAL TO GLOBAL COORDINATE SYSTEM
Considering a cross-anisotropic, linear elastic soil with isotropic properties in a general (x', y', z') plane with dip angle α to the horizontal as shown in Fig. 4, the strains are related to the stresses in the local (x', y', z') coordinate system through the relation

\[
\begin{pmatrix}
\varepsilon_{x'x'} \\
\varepsilon_{y'y'} \\
\varepsilon_{z'z'}
\end{pmatrix} =
\begin{pmatrix}
\frac{1}{E_2} - \frac{v_2}{E_2} - \frac{v_1}{E_1} & 0 & 0 \\
-\frac{v_2}{E_2} & \frac{1}{E_2} - \frac{v_2}{E_2} & 0 \\
-\frac{v_1}{E_1} & 0 & \frac{1}{G_2}
\end{pmatrix}
\begin{pmatrix}
\sigma_{x'x'} \\
\sigma_{y'y'} \\
\sigma_{z'z'}
\end{pmatrix}
\]

The local material compliance matrix \( C_{x'y'z'} \) is transformed into the global compliance matrix \( C_{xyz} \) as shown below

\[
C_{xyz} = R^T C_{x'y'z'} R
\]

where \( R \) is the transformation matrix

\[
R =
\begin{pmatrix}
cos^2 \alpha & \sin^2 \alpha & 0 & 0 & 0 & \sin 2\alpha \\
\sin^2 \alpha & \cos^2 \alpha & 0 & 0 & 0 & -\sin 2\alpha \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \cos \alpha & \sin \alpha & 0 \\
0 & 0 & 0 & -\sin \alpha & \cos \alpha & 0 \\
0.5 \sin 2\alpha & -0.5 \sin 2\alpha & 0 & 0 & 0 & \cos 2\alpha
\end{pmatrix}
\]

APPENDIX 2: CALCULATION OF CORRECTIVE STRESSES INTEGRALS
The integral of the tractions along the free surface (equation 25(d)) after some manipulation reduces to

\[
f_z(s) = 2 \lambda_1 \Phi_1(s - \lambda_1 H) + \lambda_1 \Phi_1(s - \lambda_1 H) + \lambda_2 \Phi_2(s - \lambda_2 H) + \lambda_2 \Phi_2(s - \lambda_2 H)
\]

\( s \in \mathbb{R} \)

(33)

The calculation of the stress functions of the corrective stresses \( F_1(z_1), F_2(z_2) \) requires the calculation of the infinite integral (equation (25a)),

\[
\Phi_f(z_1) = \frac{1}{\lambda_1 - \lambda_2} \int_{-\infty}^{\infty} \frac{\lambda_1 \Phi_1(\xi) + \lambda_2 \Phi_2(\xi)}{\xi - z_1} d\xi
\]

\[
\Phi_f(z_1) = \frac{1}{\lambda_1 - \lambda_2} \int_{-\infty}^{\infty} \frac{\lambda_1 \Phi_1(\xi) + \lambda_2 \Phi_2(\xi)}{\xi - z_1} d\xi
\]

\[
\Phi_f(z_1) = \frac{1}{\lambda_1 - \lambda_2} \int_{-\infty}^{\infty} \frac{\lambda_1 \Phi_1(\xi) + \lambda_2 \Phi_2(\xi)}{\xi - z_1} d\xi
\]

\[
\sum_{k=0}^{2} \int_{-\infty}^{\infty} \frac{\lambda_1 \Phi_1(\xi - \lambda_1 H) + \lambda_2 \Phi_2(\xi - \lambda_2 H)}{\xi - z_1} d\xi + \int_{-\infty}^{\infty} \frac{\lambda_1 \Phi_1(\xi - \lambda_1 H) + \lambda_2 \Phi_2(\xi - \lambda_2 H)}{\xi - z_1} d\xi
\]

(34)

Consider the integrals of the complex functions \( \Phi_k(w)/(w-z) \), \( \Phi_k(w)/(w-z) \) along the path shown in Fig. 18. This path includes branch points for function \( F_k \)

\[
w_{1,2} = \lambda_k H \pm R \sqrt{1 + \lambda_k^2}
\]

(35)

For small ratios \( R/H \), and usual degrees of anisotropy, the two branch points of \( F_k \) will lie in the upper plane (i.e. outside the chosen integration path), and therefore the integral of the analytic function \( F_k \) according to the Cauchy integral formula assumes the value

\[
\int_{w-z} \Phi_k(w) dw = 2\pi i \Phi_k(z)
\]

(36)

Also

\[
\int_{w-z} \Phi_k(w) dw = 0
\]

(37)

The final result is

\[
\Phi_f(z_1) = \frac{2}{\lambda_1 - \lambda_2} \left[ \lambda_1 \Phi_1(z_1 - \lambda_1 H) + \lambda_2 \Phi_2(z_1 - \lambda_2 H) \right]
\]

(38a)

Similarly,

\[
\Phi_f(z_2) = -\frac{2}{\lambda_1 - \lambda_2} \left[ \lambda_1 \Phi_1(z_2 - \lambda_1 H) + \lambda_2 \Phi_2(z_2 - \lambda_2 H) \right]
\]

(38b)

NOTATION

- \( a_n \): Laurent series coefficients
- \( b_n \): Laurent series coefficients
- \( C \): Integration path
- \( E_h \): Young’s modulus in (any) horizontal direction (plane of isotropy)
- \( E_v \): Young’s modulus in vertical direction
- \( E_i \): Young’s modulus in direction parallel to isotropic plane
- \( E_2 \): Young’s modulus in direction normal to isotropic plane
- \( F \): Airy’s stress function

Fig. 18. Integration path
EFFECT OF ANISOTROPY IN GROUND MOVEMENTS CAUSED BY TUNNELLING 19

\( f_s(x) \): integral of traction along boundary

\( G_{hb} \): shear modulus for strain in horizontal plane

\( G_{vb} \): shear modulus for strain in (any) vertical plane (planes of anisotropy)

\( G_z \): shear modulus for strain in direction normal to isotropic plane

\( H \): depth to tunnel springline

\( i \): imaginary unit

\( k \): hydraulic conductivity

\( K \): empirical parameter related to settlement trough width

\( K_0 \): coefficient of lateral earth pressure at rest

\( L \): radius of integration path

\( LSS^k \): constrained least-squares solution that fits \( u_0^k \)

\( N \): overload factor

\( N(x) \): normal traction on free surface

\( n, m \): stiffness ratios

\( \rho_s, q_s \): analytic coefficients

\( p_0 \): pressure outside tunnel cavity

\( p_i \): pressure inside tunnel cavity

\( R \): radius of tunnel

\( R_p \): radius of plastic zone

\( s_{ \alpha \beta } \): boundary in \( z_\alpha \) domain

\( s \): variable

\( s_u \): undrained shear strength

\( T(x) \): shear traction on free surface

\( U, V \): full-space solution (horizontal and vertical displacements)

\( u_\alpha \): horizontal ground displacements

\( u_{\alpha i} \): horizontal ground displacement measured at point \( i \)

\( u_i \): vertical ground displacements

\( u_{\alpha i} \): vertical ground displacement measured at point \( i \)

\( u_c^\alpha \): centreline surface settlement

\( u_0 \): ovalisation parameter

\( u_{0i} \): uniform convergence parameter

\( \Delta V_i/\Delta V_0 \): volume loss at tunnel cavity

\( \Delta V_i \): volume loss at ground surface

\( w_3 \): branch points of \( \phi(w) \)

\( x \): distance from tunnel centreline

\( (x, y, z) \): global coordinate system

\( (x', y', z') \): local coordinate system

\( y \): depth measured from ground surface

\( z, z_\alpha \): complex parameters

\( \alpha \): dip angle of plane with isotropic properties

\( \beta_{\alpha} \): coefficients related to stiffness parameters

\( \gamma_i \): shear strain

\( \epsilon_{\alpha i} \): normal strain

\( \xi_\alpha \): transformed variable

\( \theta \): angle

\( \lambda_k \): roots of the characteristic equation (with positive imaginary part)

\( \psi \): Poisson’s ratio (isotropic case)

\( v_{\alpha h} \): Poisson’s ratio for effect of horizontal strain on complementary horizontal strain

\( v_{\alpha v} \): Poisson’s ratio for effect of horizontal strain on vertical strain

\( v_{\beta h} \): Poisson’s ratio for effect of vertical strain on horizontal strain

\( v_1 \): Poisson’s ratio for effect of strains in isotropic plane \((x’–z')\)

\( v_2 \): Poisson’s ratio for effect of strain in \( y' \) direction due to strain in \( x' \) direction

\( \xi \): integration variable

\( \rho \): ovalisation ratio

\( \sigma_i \): analytic coefficient

\( \sigma_{0b} \): normal stress

\( \sigma_{0s} \): overburden pressure

\( \Sigma_\alpha \): boundary in \( \xi_\alpha \) domain

\( \tau_0 \): shear stress

\( \Phi_\lambda(z) \): analytic function

Superscripts

\(+\): corresponding to cavity at \((0, H)\)

\(-\): corresponding to cavity at \((0, -H)\)

\(c\): ‘corrective’ solutions

Subscripts

\(B\): boundary

\(k\): integer (assumes values 1, 2)

REFERENCES


Pinto, F. (1999). Analytical methods to interpret ground deformations due to soft ground tunneling. SM thesis, Department of Civil & Environmental Engineering, MIT, Cambridge, MA, USA.


